Semiclassical analysis of defect sine-Gordon theory

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Abstract

The classical sine-Gordon model is a two-dimensional integrable field theory, with particle like solutions the so-called solitons. Using its integrability one can define its quantum version without the process of canonical quantization. This bootstrap method uses the fundamental propterties of the model and its quantum features in order to restrict the structure of the scattering matrix as far as possible. The classical model can be extended with integrable discontinuities, purely transmitting jump-defects. Then the quantum version of the extended model can be determined via the bootstrap method again. But the outcoming quantum theory contains the so-called CDD uncertainity. The aim of this article is to carry throw the semiclassical approximation in both the classical and the quantum side of the defect sine-Gordon theory. The CDD ambiguity can be restricted by comparing the two results. The relation between the classical and quantum parameters as well as the resoncances appeared in the spectrum are other objectives.

1 Introduction

Recently, there has been an increasing interest in integrable quantum field theories including defects or impurities. The motivations come from both the theoretical and the application part of physics. In the solid state and in the statistical physics there are realistic physical applications. There is also a need of theoretical understanding of this kind of field theory, which gives a well defined model for impurities.

Due to the no-go theorem formulated by Delfino, Mussardo and Simonetti the defect theories stay in the background for some time [1], [2]. The theorem states that a relativistically invariant theory with an integrable non-free interaction in the bulk region permits only two types of defects: the purely transmitting and the purely reflecting ones. This theorem was originally formulated for diagonal theories, and later it was extended to many type of non-diagonal ones [3]. (Although some effort has been made to overcome this obstacle by giving up the Lorentz invariance, see for instance [4] and references therein).

As the no-go theorem showed non-free integrable defect theories are purely transmitting. This proof retains the researchers for some time to analyze these models. But the subject were revived after founding explicit Lagrangian realizations for the defect theories [6]. Following the original idea many integrable defect theories were constructed at the classical level [7], [8], [9]. The basis for the quantum formulation of defect theories is provided by the folding trick [10] by which one can map any defect theory into a boundary one. As a consequence defect unitarity, defect crossing symmetry and defect bootstrap equations together with defect Coleman-Thun mechanism are derived. Despite of these results the explicitly solved relativistically invariant defect quantum field theories are quite rare, containing basically the sine-Gordon and affine Toda field theories [11], [12], [13]. The explicit solution of the Lee-Yang and sinh-Gordon theories are also known [14].

It is required that the transmission matrix of the defect theory satisfy the Yang-Baxter equation (also known as "factorisability condition"), and the standard equation of unitarity and crossing symmetry. These equations have enough restrictive power to determine the T-matrix up to the so-called "CDD ambiguity".

It means that there is an unknown scalar function in the scattering matrix and - a different one - in the defect transmission matrix. These functions cannot be fixed via the bootstrap method. One way to restrict this ambiguity is to calculate the phase shift by the semiclassical approximation. In the case of quantum sine-Gordon theory the classical phase shift has been compared with the scattering matrix earlier [16], [17]. One of the aims of this article is to determine the phase shift in the case of *defect* sine-Gordon theory. Using the result we can anchor the CDD uncertainty. The defect theory has new symmetries, so we have to recalculate the time delay for the new cases.

Another objective is to map the pole structure of the T-matrix. In this way we can study the spectrum of the theory. This direct examination can show if there are stable or unstable bound states or resonances in the model.

This paper is organized as follows: In section 2, we explain the classical defect sine-Gordon theory, where we give the time delay for the case of odd topological charge. In section 3 we introduce the defect quantum sine-Gordon theory and outline the semiclassical approximation. In section 4 we perform the semiclassical approximation on the basis of the previous sections. Firstly we start from the classical side. Then we calculate the phase shift from the transmission matrix in the case of the even and odd charged defects. We analyze the matrix structure in the same limit which lead us to analyze the spectrum directly in section 5. Finally we conclude in section 6 and give directions for further research.

2 Classical defect sine-Gordon theory

In the bulk the sine-Gordon model can be defined by the following Lagrangian density:

$$\mathcal{L}_{\Phi} := \frac{1}{2}\left((\partial_t\Phi)^2 - (\partial_x\Phi)^2\right) + \frac{m^2}{\beta^2}(\cos(\beta\Phi) - 1) \tag{1}$$

The integrability follows from the continuity equations for an infinite set of local currents. For classical considerations it is often convenient to remove the mass parameter m and the coupling β by rescaling the field. The Lagrangian is invariant under the well known bulk symmetries

$$\Phi \to \Phi + \frac{2n\pi}{\beta},\ n \in \mathbb{Z} \tag{2}$$

, which interpolate among the different ground states of the bulk.

A single jump-defect placed at $x = 0$ is described by modifying the Lagrangian. Firstly we define two fields: the u field for the $x < 0$ and v for the $x > 0$ part of the x-axis. The "time evolution" of these two restricted fields will follow the bulk region. The defect works as an internal boundary linking the two fields: a delta function contribution at $x = 0$ couples u and v. The detailed Lagrangian density is the following [15]

$$\mathcal{L} := \Theta(-x)\mathcal{L}_u + \Theta(x)\mathcal{L}_v - \delta(x)\left[\frac{1}{2}\left(u\partial_t v - v\partial_t u\right) - \mathcal{B}(u,v)\right] \tag{3}$$

where the defect potential is

$$\mathcal{B} = -\frac{2m\sigma}{\beta^2}\cos\left(\beta\frac{u+v}{2}\right) - \frac{2m}{\sigma\beta^2}\cos\left(\beta\frac{u-v}{2}\right) \tag{4}$$

It follows that the bulk field equations - after the suitable rescaling - for u and v are:

$$\begin{aligned} x < 0 &: \ \partial_\mu\partial^\mu u = -\sin(u) \\ x > 0 &: \ \partial_\mu\partial^\mu v = -\sin(v) \end{aligned} \tag{5}$$

For the $x=0$ point we get the defect conditions:

$$\begin{array}{c} \partial_x u - \partial_t v = -\sigma \sin\left(\frac{u+v}{2}\right) - \frac{1}{\sigma}\sin\left(\frac{u-v}{2}\right) \\ \partial_x v - \partial_t u = \sigma \sin\left(\frac{u+v}{2}\right) - \frac{1}{\sigma}\sin\left(\frac{u-v}{2}\right) \end{array} \tag{6}$$

The first order time derivatives in the defect Lagrangian is required by integrability. The theory has an infinite set of mutually commuting integrals of motion.

The period of the defect potential is twice of the bulk potential. So the new Lagrangian is not invariant under the bulk symmetries $u \rightarrow u + 2a\pi/\beta$, $v \rightarrow v + 2b\pi/\beta$ where a and b are integers, The new symmetries are

$$u \rightarrow u + 4a\pi/\beta, v \rightarrow v + 4b\pi/\beta,\ a, b \in \mathbb{Z} \tag{7}$$

It is not even invariant under the reflections $u \rightarrow -u$ or $v \rightarrow -v$. But (3) is invariant under certain combinations of the earlier transformations, such as reflecting both fields simultaneously.

The Lagrangian does not violate time translation. It follows that the total energy $\mathcal{E}$ will be conserved. In a certain time this constant $\mathcal{E}$ consists of the energy of both fields, and a contribution from the defect. This contribution is a function of the fields evaluated at the place of the defect

$$\mathcal{E} = E\left(u\right) + E\left(v\right) + B\left(u, v\right) \tag{8}$$

A single soliton of the sine-Gordon model may be described by

$$u(t, x) := \pm\frac{4}{\beta}\arctan\left(e^{-m\frac{x+vt}{\sqrt{1-v^2}}}\right) \tag{9}$$

where $+$ stands for solitons, and $-$ for anti-solitons, and the v quantity is the velocity of the soliton. The energy of this soliton from the Hamilton density

$$E = \frac{M}{\sqrt{1-v^2}} = M\cosh(\theta) \tag{10}$$

where θ is the rapidity, $M = 8m/\beta^2$.

It is useful to write the above solution in the following - equivalent - form [15]

$$e^{iu/2} = \frac{1 \pm iE}{1 \mp iE},\ E = e^{ax+bt+c} \tag{11}$$

where in terms of rapidity $a = \cosh(\theta), b = -\sinh(\theta)$ are real quantities and clearly $a^2 - b^2 = 1$. Now the upper sign stands for solitons. Notice that there is no reason why $u = v$ at $x = 0$, that is the whole field is not necessarily continuous.

Suppose there is a soliton moving in a positive sense along the x-axis described by the u field. It encounters the defect, then a similar, but delayed, v field soliton emerges. We suppose that the solution for the v field

$$e^{iv/2} = \frac{1 + ziE}{1 - ziE},\ E = e^{ax+bt+c} \tag{12}$$

where z contains the time delay in this manner

$$zE = e^{ax+bt+c+ln(z)} \Rightarrow \Delta t = \frac{ln(z)}{b} \tag{13}$$

If we rewrite the defect parameter $\sigma = e^{-\eta}$, the expression for z is the following

$$z = \frac{e^{-\theta}+\sigma}{e^{-\theta}-\sigma} = \coth\left(\frac{\eta-\theta}{2}\right) \tag{14}$$

which can be substituted into (13). The time delay, which was calculated already in [15]

$$\Delta t = \frac{\ln\left(\tanh\left(\frac{\eta-\theta}{2}\right)\right)}{\sinh(\theta)} \tag{15}$$

Suppose that $\theta > 0$ and $\eta < 0$. In this case $z < 0$, which means that the incoming soliton flips to an anti-soliton. But in the $\eta > 0$ case there are several possibilities. If $\theta < \eta$ the type of soliton remains unchanged; if $\theta > \eta$ the soliton type changes on the defect. In the special $\theta = \eta$ case the incoming soliton is delayed infinitely, in effect the incoming soliton is bounded. In this situation the incoming soliton can be replaced in the far future by the static $u = 0$, $v = 2\pi$ solution. This solution stores the same energy and momentum into the defect as the incoming soliton has, so the defect works like a capacitor in an electrical circuit.

We can generalize this idea, since the two static fields $u = 2\pi n$ and $v = 2\pi m$ satisfies the (5) equations of motion and the (6) defect conditions if n and m are integers. The conserved topological charge is

$$Q := \frac{\beta}{2\pi}\int\limits_{-\infty}^{\infty} dx\partial_x\phi = \frac{\beta}{2\pi}(\lim_{x\to+\infty} v(x) - \lim_{x\to-\infty} u(x) + \lim_{x\to+0} v(x) - \lim_{x\to-0} u(x)) = n - m$$

where the limit at the origin measures the strength of the defect [6].

We have to discuss the time delay in a new unnamed case : we have to take care of the parity of the charge, because of the new (7) symmetries. To show this let $v \to v + 2\pi$, while u remains unchanged

$$e^{iv/2} = -\frac{1+ziE}{1-ziE} \tag{16}$$

In this case the defect links solitons differing by one unit of charge. Suppose the situation $ax + bt + c = 0$ so $E = 1$. Substituting this solution into the defect conditions the equation for z is

$$\frac{4a}{2} - \frac{4bz}{1+z^2} = \frac{1}{2}i\sigma\left(\frac{(1+i)(1+iz)}{(1-i)(1-iz)} - \frac{(1-i)(1-iz)}{(1+i)(1+iz)}\right) + \frac{1}{2}\frac{i}{\sigma}\left(\frac{(1+i)(1-iz)}{(1-i)(1+iz)} - \frac{(1-i)(1+iz)}{(1+i)(1-iz)}\right) \tag{17}$$

After some simplification we get

$$z = \tanh\left(\frac{\eta - \theta}{2}\right) \tag{18}$$

The corresponding time delay is negative of that of the even case

$$\Delta t = -\frac{\ln(\tanh(\frac{\eta-\theta}{2}))}{\sinh(\theta)} \tag{19}$$

There are N-soliton type solutions in the bulk, which can be generated by Hirota's method. In this article we use only the two-soliton solutions. A soliton-antisoliton pair - moving with the relative velocity v - can be written

$$\Phi_{s\overline{s}} = \frac{4}{\beta}\arctan\left(\frac{\sinh(m\gamma vt)}{v\cosh(m\gamma x)}\right),\ \gamma = 1/\sqrt{1-v^2} \tag{20}$$

From this expression we can easily obtain the well known time delay

$$\Delta t = \frac{2ln(v)}{m\gamma v} = \frac{2\ln(\tanh(\theta))}{m\sinh(\theta)},\ \theta = \theta_2 - \theta_1 \tag{21}$$

with the relative θ rapidity. The presence of the time delay shows that there is an interaction between solitons.

Worthy of note that the (21) time delay is exactly twice of the (15) time delay on the defect (if formally we think of the η defect parameter as one of the rapidity).

3 Quantum defect sine-Gordon theory

In the quantum theory of the sine-Gordon model the $\mathcal{H}$ Hilbert-space is the Fock-space of multiparticle states. The vectors of $\mathcal{H}$ is generated "particle creation operators" $A_a(\theta)$

$$|A_{a_1}(\theta_1)\, A_{a_2}(\theta_2)\, ...A_{a_n}(\theta_n)\rangle = A_{a_1}(\theta_1)\, A_{a_2}(\theta_2)\, ...A_{a_n}(\theta_n)\,|0\rangle \tag{22}$$

These vectors can be interpreted as the asymptotic ("in-" or "out-") scattering states [5], which correspond to the limit of the soliton type solutions in the far past and in the far future. In the defect theory we can describe the purely transmitting defect with a matrix which relates these "in-" and "out-" vectors of $\mathcal{H}$.

We could see using (15) that a classical defect can delay a soliton or an anti-soliton infinitely and the final state might be replaced by a static one. In the corresponding quantum theory we have to say the defect "bounds" this particle. Reversing the sense of time it appears that defects can decay and might produce particles resemblance to an excited atom. It is clear that an incoming soliton can induce a decay.

We have to take into account also that in the defect theory the bulk symmetries are broken. This idea lead us to distinguish the even and odd charged defects. In the corresponding quantum theory we have to introduce different transfer matrices for them.

Finally, the notation for the even type will be [15]

$$ {}^{e}T_{a\alpha}^{b\beta}(\theta) \tag{23} $$

where a labels the incoming particle and b the outgoing one. Its value may be $+$ for a soliton or $-$ for an anti-soliton. The α and β numbers are even integer; they sign the topological charge of the in and out particles and $\beta-\alpha$ is the even defect charge. This matrix regarded as describing the transmission of a particle with rapidity θ from the $x<0$ region to the region $x>0$. The structure of this matrix can be explored via the bootstrap method.

Firstly, these families of even transmission matrices should satisfy the unitarity condition (for real rapidity)

$$ {}^{e}T(\theta){}^{e}T^{\dagger}(\theta) = 1 \tag{24} $$

The transmission matrices relate states of the system in the far future to those in the far past and the state of the system is labelled by the soliton rapidity and its topological charge together.

We have to invoke the well known sine-Gordon S-matrix for a two-body scattering process in the bulk [20]. It is given by

$$ S_{kl}^{mn}(\theta) := \varrho(\theta) \begin{pmatrix} a(\theta) & 0 & 0 & 0 \\ 0 & c(\theta) & b(\theta) & 0 \\ 0 & b(\theta) & c(\theta) & 0 \\ 0 & 0 & 0 & a(\theta) \end{pmatrix} \tag{25} $$

where k,l label the incoming particles and m,n the outgoing ones. $\theta=\theta_1-\theta_2$ where the θ_1 rapidity belong to the k,n and θ_2 is the rapidity of the l,m particles. The various pieces of the matrix are defined by:

$$ a(\theta) := e^{i\pi\gamma}e^{-\gamma\theta} - e^{-i\pi\gamma}e^{\gamma\theta},\ b(\theta) := e^{\gamma\theta} - e^{-\gamma\theta},\ c(\theta) := e^{i\pi\gamma} - e^{-i\pi\gamma} \tag{26} $$

In this notation the crossing property of the S-matrix is represented by

$$ S_{kl}^{mn}(i\pi-\theta) = S_{k\ -m}^{-l\ n}(\theta) \tag{27} $$

The overall $\varrho\,(\theta)$ function is

$$ \varrho(\theta) := \frac{\Gamma(1+i\gamma\theta/\pi)\Gamma(1-\gamma-i\gamma\theta/\pi)}{2\pi i} \prod_{k=1}^{\infty} R_k(\theta)R_k(i\pi-\theta) \tag{28} $$

where

$$ R_k(\theta) = \frac{\Gamma(2k\gamma+i\gamma\theta/\pi)\Gamma(1+2k\gamma+i\gamma\theta/\pi)}{\Gamma((2k+1)\gamma+i\gamma\theta/\pi)\Gamma(1+(2k-1)\gamma+i\gamma\theta/\pi)} \tag{29} $$

The conventions adopted by Konik and LeClair [11] have been used. Therefore the coupling γ in terms of the Lagrangian coupling β is given by

$$\frac{1}{\gamma}=\frac{\beta^2}{8\pi-\beta^2} \tag{30}$$

Since the defect is purely transmitting, the heuristic arguments based on factorisability and bulk integrability would require [1]

$$S_{kl}^{mn}(\theta)\,{}^eT_{n\alpha}^{t\beta}(\theta_1)\,{}^eT_{m\beta}^{s\gamma}(\theta_2)=\ {}^eT_{l\alpha}^{n\beta}(\theta_2)\,{}^eT_{k\beta}^{m\gamma}(\theta_1)S_{mn}^{st}(\theta) \tag{31}$$

It is useful to structure the transmission matrix into block matrices. The entries are infinite dimensional block matrices labelled by the topological charge of the defect

$${}^eT=\begin{pmatrix} T_+^+ & T_+^- \\ T_-^+ & T_-^- \end{pmatrix}\equiv\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{32}$$

Using the triangle relation (31) and a collection of general principles they following result can be obtained [15]

$$\begin{pmatrix} A_\alpha^\beta & B_\alpha^\beta \\ C_\alpha^\beta & D_\alpha^\beta \end{pmatrix}:=f(q,x)\begin{pmatrix} \frac{1}{\sqrt{\nu}}Q^\alpha\delta_\alpha^\beta & e^{-i\pi\gamma/2+\gamma(\theta-\eta)}\delta_\alpha^{\beta-2} \\ e^{-i\pi\gamma/2+\gamma(\theta-\eta)}\delta_\alpha^{\beta+2} & \sqrt{\nu}Q^{-\alpha}\delta_\alpha^\beta \end{pmatrix} \tag{33}$$

The definition of the overall f function is given by

$$f(q,x):=\frac{e^{i\pi(1+\gamma)/4}}{(1+ipx)}\frac{r(\theta)}{r(-\theta)},\quad px:=e^{\gamma(\theta-\eta)} \tag{34}$$

where

$$r(x)=\prod_{k=0}^{\infty}\frac{\Gamma(k\gamma+1/4-i\gamma\tilde{\theta}/2\pi)\Gamma((k+1)\gamma+3/4-i\gamma\tilde{\theta}/2\pi)}{\Gamma((k+1/2)\gamma+1/4-i\gamma\tilde{\theta}/2\pi)\Gamma((k+1/2)\gamma+3/4-i\gamma\tilde{\theta}/2\pi)},\tilde{\theta}:=\theta-\eta \tag{35}$$

The examination of the poles of ${}^eT$ indicates that some pole corresponds to an unstable defect bound state. For this reason, there will be a bootstrap condition linking the the even and odd transmission matrices. In detail

$$c_{b\alpha}^{\gamma}\,{}^oT_{a\gamma}^{c\delta}=S_{ab}^{pq}\left(\theta-\eta+\frac{i\pi}{2\gamma}\right){}^eT_{q\alpha}^{c\beta}c_{p\beta}^{\delta} \tag{36}$$

For the odd case we use the notation $\hat{A}$ in place of A. From the bootstrap condition we get

$$\hat{A}(\theta)=\frac{1}{\sqrt{\nu}}\frac{e^{-i\pi(1+\gamma)/4}}{1+ipx}\frac{\cos(\pi/4\gamma-i(\theta-\eta)/2)}{\sin(\pi/4\gamma-i(\theta-\eta)/2)}\frac{s(x)}{\overline{s}(x)} \tag{37}$$

where $s(x)$ is very similar to $r(x)$

$$s(x)=\prod_{k=0}^{\infty}\frac{\Gamma(k\gamma+3/4+i\gamma\tilde{\theta}/2\pi)\Gamma((k+1)\gamma+1/4+i\gamma\tilde{\theta}/2\pi)}{\Gamma((k+1/2)\gamma+1/4+i\gamma\tilde{\theta}/2\pi)\Gamma((k+1/2)\gamma+3/4+i\gamma\tilde{\theta}/2\pi)} \tag{38}$$

It is clear that the *matrix equation*s above do not have a unique solution. One can multiply a given solution with a proper scalar function factor and the result remains a solution. This scalar function must satisfy the unitarity condition and the crossing property

$$\begin{aligned} f(\theta)f(-\theta) &= 1 \\ f(i\pi-\theta) &= f(\theta) \end{aligned}$$

This undetermined scalar function is the referred "CDD ambiguity".

4 Semiclassical comparison

Suppose that Ψ is the wave function of the *in-state* which is given in the (22) form. The "defect" theory and the "bulk" theory yield different out-states because of the interaction with the defect. Now we turn to calculate the shifted phase in the "defect" theory from the classical and from the quantum model. Then we compare the results to restrict the scalar function of the "CDD ambiguity" in question.

4.1 Starting from the classical side

To calculate the phase shift from the classical model we use the semiclassical approximation. The semiclassical method is worked out for particle quantum mechanics but it can be generalized for the field theory. In this paper the method of Jackiw and Woo will be followed [16].

Let's write the Schrödinger equation

$$\hat{H}\Psi = E\Psi \tag{39}$$

then make the usual assumption that the wave function has the following separate form

$$\Psi(x,t) = A(x,t)e^{\frac{i}{\hbar}S(x,t)}, \tag{40}$$

The most important guess: in space and time the amplitude $A(x,t)$ changes much more slowly than the phase $S(x,t)/\hbar$. At first look it seems that this corresponds to the limit $\hbar \to 0$. But - following from the series expansion - Ψ has a relevant singularity at this point. So it is better to say that $S \gg \hbar$, or - in field theory - the coupling constant is supposed to be very small.

Using (39) we get in the leading order

$$S(x) = \int_{x_0}^{x_f} p(x,E)dx + \hbar\varphi_0 \tag{41}$$

where $p(x,E)$ is the momentum affected by the presence of the potential. The known result for the semiclassical phase shift

$$2\delta(E) = \lim \int_{x_0}^{x_f} dx(p(x,E) - p(E)) \tag{42}$$

But starting from the Lagrangian one can derive a connection between the time delay and the momentum of the *classical* particle

$$\Delta t\,(E) = \frac{\partial}{\partial E}\int_{x_0}^{x_f} dx(p(x,E) - p(E)) \tag{43}$$

where $p(E)$ is the momentum of the free particle. This is the time of flight for a classical particle moving from position x_0 to x_f. In the case of unbounded motion we have to take the limit $x_0 \to \infty, x_f \to \infty$.

Comparing the last two formulas the important result is the following

$$\delta(E) - \delta(E_{th}) = \frac{1}{2}\int_{E_{th}}^{E} dE' \triangle t(E')$$

Jackiw and Wo generalize this idea for the field theory: the formula remains unchanged but the x_0 and x_f is replaced by the Φ_i initial and Φ_f final field configurations. The time of flight means the time of field evolution between the Φ_i and Φ_f configurations in the presence of a potential. The phase shift formula for the *field* theory [16]

$$\delta(E) = n_B\pi + \int_{E_{th}}^{E} dE' \triangle t(E') \tag{44}$$

In this formula the lower bound of the integral E_{th} has a meaning : it is the threshold energy where $p\,(E_{th}) = 0$. Below the threshold energy the time delay is infinite, so the field cannot reach the final stage. Hence we are interested in the case when the integral starts *just above* E_{th}. The bound states start somewhere just below E_{th}. Its phase shift contribution is taken into account with n_B the number of bound states.

Now we only have to substitute the time delay (15) into the previous formula (44). To do this one must change the variable of integration from the energy E to the rapidity θ, and we need an expression for

$$\frac{\partial E\,(\theta)}{\partial\theta} \tag{45}$$

The (10) formula is the energy of one soliton in the "bulk". In the defect case the energy is conserved, hence the whole energy can be evaluated at every position of the soliton. Suppose that the soliton is far from the defect. In this case the energy stored in the defect is a constant value: independent from θ. So in the (45) derivative we can use the formula for the "bulk".

The hyperbolic function in the denominator of the (15) time delay disappears because

$$\frac{\partial E}{\partial\theta} = M\sinh(\theta) \tag{46}$$

After the substitution the result from the classical side

$$\delta(E) = n_B\pi + \frac{8m}{\beta^2}\int_0^{\theta} d\theta' \ln\left[\tanh\frac{(\eta-\theta')}{2}\right] \tag{47}$$

4.2 Starting from the quantum side

Now we turn to the quantum field theory, where we know the even transmission matrix (32). The matrix entries, especially the common (34) f function, are product of gamma functions. Accordingly to extract the phase shift we use an integral representation of the natural logarithm of gamma functions

$$\ln(\Gamma(\zeta)) = \int_0^\infty \frac{dt}{t} e^{-t} \left[\zeta - 1 + \frac{e^{-(\zeta-1)t} - 1}{1 - e^{-t}} \right], \ \mathfrak{Re}(\zeta) > 0 \tag{48}$$

To be explicit we calculate the $A = T_+^+$ element by using its logarithm. Then the logarithm of infinite products turns into infinite summations, and we can use the (48) formula.

After some calculation we reach a formula, which can be rearranged in terms of trigonometric and hyperbolic functions

$$A(\theta) = \frac{1}{\sqrt{\nu}} \frac{e^{i\pi(1+\gamma)/4}}{(1+ipx)} \exp\left(i \int_0^\infty \frac{dt}{t} \frac{\sin\left(\frac{\gamma\tilde{\theta}t}{\pi}\right) \sinh\left(\frac{t}{2}(\gamma+1)\right)}{\sinh(t)\cosh\left(\frac{\gamma t}{2}\right)} \right) \tag{49}$$

If we want to separate the φ phase shift we have to concentrate on the argument of the exponential in the form $e^{i\varphi}$. The semiclassical approximation means that in the integrand we substitute the component functions with its Taylor-series. Then we keep only the leading order in the γ coupling constant.

After making this process the resulting expression for the phase shift

$$\varphi = \int_0^\infty \frac{dt}{t^2} \sin((\theta - \eta)t) \tanh\left(\frac{1}{2}\pi t\right) \frac{\gamma}{\pi} \tag{50}$$

We can use the (30) connection between the coupling constants in order to express γ with β in the leading order

$$\frac{1}{\gamma} = \frac{\beta^2}{8\pi} \left(\frac{1}{1 - 8\pi/\beta^2} \right) = \frac{\beta^2}{8\pi} \left(1 + \frac{8\pi}{\beta^2} + \left(\frac{8\pi}{\beta^2}\right)^2 + ... \right) \tag{51}$$

In the leading order it follows that

$$\gamma = \frac{8\pi}{\beta^2}$$

Then we can express (50)

$$\varphi = \frac{8}{\beta^2} \int_0^\infty \frac{dt}{t^2} \sin((\theta - \eta)t) \tanh\left(\frac{1}{2}\pi t\right) + \mathcal{O}(\beta^\circ) \tag{52}$$

4.3 Comparison of the two results

The final equation in question come from (47) and (50):

$$\int_0^\infty \frac{dt}{t^2} \sin\left((\theta-\eta)t\right) \tanh\left(\frac{1}{2}\pi t\right) = \int_0^\theta d\theta' \ln\left(\tanh\frac{(\eta-\theta')}{2}\right) \tag{53}$$

Differentiating both sides with respect to θ:

$$\int_0^\infty \frac{dt}{t} \cos\left((\theta-\eta)t\right) \tanh\left(\frac{1}{2}\pi t\right) = \ln\left(\frac{\eta-\theta}{2}\right) \tag{54}$$

This is a known identity, which can be found in table of integrals, for example in [22]. We have proven that the two semiclassical limit give the *same* result. So we can assume that the unknown CDD scalar function is the simplest one.

4.4 The case of odd charged defects

If the defect had an odd charge we have to start the calculation from the (38) expression. Using our (48) integral representation for the gammas the result for the exponential in $\hat{A}(\theta)$

$$\exp\left(i\int_0^\infty \frac{dt}{t} \frac{-\sin\left(\gamma\frac{\tilde{\theta}t}{\pi}\right)\sinh\left(\frac{1}{2}t(\gamma-1)\right)}{\sinh(t)\cosh\left(\frac{t\gamma}{2}\right)}\right) \tag{55}$$

Let us compare this formula with the earlier and very similar (49) expression which we found in the case of the even charged defect. The difference in the argument of the hyperbolic function - $+$ or $-$ sign - doesn't count in the $\gamma\to\infty$ semiclassical limit. So the new formula differs only a minus sign compared to (52)

$$\varphi = -\frac{8}{\beta^2}\int_0^\infty \frac{dt}{t^2} \sin\left((\theta-\eta)t\right) \tanh\left(\frac{1}{2}\pi t\right) + \mathcal{O}(\beta^\circ) \tag{56}$$

We have found this minus sign in the (19) time delay of the odd charged defect. In this way we can trace back the checking of the phase shifts to the even case: we found that the two result is *equal* again.

After this we examine the structure of the even transmission matrix ${}^eT$ in the semiclassical limit, which was given by (33). In this limit the quantum features of the model disappear and graduate into classical behavior. If we want unambiguous result we must fix the relative value of the parameters of motion: the rapidity of the soliton and the parallel defect parameter.

Assume that

$$\theta > \eta \Longleftrightarrow \theta - \eta > 0 \tag{57}$$

and suppose that the γ coupling constant is going to infinity. In this limit we find that the limit value of the off-diagonal element of the transmission matrix

$$\frac{e^{i\pi(1+\gamma)/4}e^{-i\pi\gamma/2+\gamma(\theta-\eta)}}{(1+ie^{\gamma(\theta-\eta)})} = -\frac{e^{-i\pi\gamma/4}e^{\gamma(\theta-\eta)}}{(1+ie^{\gamma(\theta-\eta)})} \to ie^{-i\pi\gamma/4} \tag{58}$$

It is important to note that the exponent is proportional to the coupling constant, which can be an indication that bound states or resonances exists in the spectrum. In the classical case using (15) we saw that there is no upper bound of the time delay. But it doesn't mean necessarily that there are bound states in the quantum version. If we consider the phenomenon of quantum tunnelling maybe we can find only short life resonances.

In the same limit the diagonal elements of the matrix goes to zero:

$$\left|\frac{e^{i\pi(1+\gamma)/4}}{(1+ie^{\gamma(\theta-\eta)})}\right| = \frac{1}{\left|1+ie^{\gamma(\theta-\eta)}\right|} \leq \frac{1}{\left|e^{\gamma(\theta-\eta)}\right|} \to 0, \quad \gamma \to \infty \tag{59}$$

So the matrix has the following symbolic structure in the semiclassical limit

$$\begin{pmatrix} 0 & \bullet \\ \bullet & 0 \end{pmatrix} \tag{60}$$

which shows that in this case the type of the soliton always changes, when it goes over the defect.

Reverse the relation between the parameters, that is to say

$$\theta < \eta \Longleftrightarrow \theta - \eta < 0 \tag{61}$$

In this case the situation changes, the off-diagonal elements go to zero

$$\left|\frac{e^{i\pi(1-\gamma)/4}\overbrace{e^{\gamma(\theta-\eta)}}^{\to 0}}{(1+i\underbrace{e^{\gamma(\theta-\eta)}}_{\to 0})}\right| \to \left|\frac{e^{i\pi(1-\gamma)/4}0}{(1+0)}\right| = 0 \tag{62}$$

and the diagonal elements have the finite value:

$$\frac{e^{i\pi(1+\gamma)/4}}{(1+ie^{\gamma(\theta-\eta)})} = -e^{i\pi\gamma/4} \tag{63}$$

The structure we found for the transmission matrix

$$\begin{pmatrix} \bullet & 0 \\ 0 & \bullet \end{pmatrix} \tag{64}$$

which represents that the soliton type remain unchanged.

5 Resonances

The analysis of the structure of the even transmission matrix ${}^{e}T$ shows that bound states or resonances can appear in the spectrum of the transmission. The limit values in the semiclassical limit were proportional to the coupling.

This phenomena can be examined directly, by extracting the pole structure of the transmission matrix. To do this we invoke that a gamma function has a simple pole at every non positive integer. But we have to take care because there are poles in denominator of the (33) formula: if the pole of the denominator is at the same location as the pole of the numerator, then the pole in question has been cancelled out.

Firstly we would try to find the bound states. This means that we have to locate the poles in the so-called "physical strip" [5]. It means that the rapidity should be in the $[0, i\pi/2)$ domain of the complex plane.

From the (34) and the expression (35) it is clear that there are four *indexed* gamma function in the numerator, and four similar in the denominator. The eight equations for the poles

$$\begin{aligned} k\gamma + \frac{1}{4} \mp \frac{i\gamma\widetilde{\theta}}{2\pi} &= -n \\ (k+1)\gamma + \frac{3}{4} \mp \frac{i\gamma\widetilde{\theta}}{2\pi} &= -n \\ (k+\frac{1}{2})\gamma + \frac{1}{4} \pm \frac{i\gamma\widetilde{\theta}}{2\pi} &= -n \\ (k+\frac{1}{2})\gamma + \frac{3}{4} \pm \frac{i\gamma\widetilde{\theta}}{2\pi} &= -n \end{aligned} \tag{65}$$

where $n = 0, 1, 2, ...$ The upper sign denotes the numerator, and the lower the denominator.

After expressing the rapidity we found that the n-th pole of the k-th gamma is at rapidity

$$\begin{aligned} \widetilde{\theta} &= \mp i\left(2\pi k + \frac{\pi}{2\gamma}(4n+1)\right) \\ \widetilde{\theta} &= \mp i\left(2\pi(k+1) + \frac{\pi}{2\gamma}(3+4n)\right) \\ \widetilde{\theta} &= \pm i\left(\pi(2k+1) + \frac{\pi}{2\gamma}(4n+1)\right) \\ \widetilde{\theta} &= \pm i\left(\pi(2k+1) + \frac{\pi}{2\gamma}(4n+3)\right) \end{aligned} \tag{66}$$

where the notation of the signs is the same. ($\widetilde{\theta} = \theta - \eta$ but η is real, so it doesn't effect the domain). Taking into account the poles of the denominator we found that there are no poles in the "physical strip".

After this we turn to the examination of so-called "resonance" poles [5]. It means that we should locate poles with rapidity in the $[0, -i\pi/2)$ interval. We can use the (66) formulas without change, together with the similar expressions for the denominator.

In this case we found poles in the given domain. In the semiclassical limit $\gamma \to \infty$ the number of poles is proportional to the coupling γ

$$n_p \sim \frac{\gamma}{4} \tag{67}$$

This relation agree with the expressions (58) and (63) found in the semiclassical limit of the even transmission matrix.

6 Conclusion

In this article an interesting version of the nonlinear sine-Gordon field theory has been examined. The theory includes an integrable jump-defect in the origin, which means that the integrability - an important characteristic of the original model - is not destroyed.

It is required that the T-matrix of the defect theory satisfy the Yang-Baxter equation (also known as "factorisability condition"), and the standard equation of unitarity and crossing symmetry. These equation have enough restrictive power to determine the T-matrix up to the so-called "CDD ambiguity".

It means that there is an unknown scalar function in the scattering and the defect transmission matrix too. This function cannot be fixed via the bootstrap method. One way to fix this ambiguity is to calculate the phase shift by the semiclassical approximation. In the case of quantum sine-Gordon theory the classical phase shift has been compared with the scattering matrix earlier [16], [17]. One of the aims of this article is to determine the phase shift in the case of *defect* sine-Gordon theory. Using the result we can anchor the CDD uncertainty. The defect theory has new symmetries so we had to recalculate the time delay for a new case.

Another objective is to map the pole structure of the T-matrix. In this way we can study the spectrum of the theory if there are stable or unstable bounds states or resonances.

In the above sections we have made the semiclassical limit, and we have found that in the leading order of the semiclassical approximation the classical and the quantum field theory agree: the phase shifts are equal. The unknown scalar function can be the trivial one.

The classical results confirmed that a defect - effectively - behaves like a half soliton: the time delay is half of time delay if the soliton scatters on an antisoliton. Naturally the result for the phase shift is the same. The similarity in the quantum version remained an open question.

We examine the structure of the transmission matrix in the semiclassical limit too. The results indicate that there can be stable or unstable bound states or resonances in the spectrum. The remaining matrix entries are the function

of the quarter of the coupling constant, which shows the half-soliton behavior again.

But the direct inspection demonstrated that there are no bound states in the spectrum of a sine-Gordon theory with a jump-defect. But there are resonances in the model and naturally the number of resonances is proportional to the coupling constant.

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